Floating Point Numbers

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Administrivia

- Midterm: Oct 9th (Wednesday) in class
- PA2 out now
- `gdb` lab

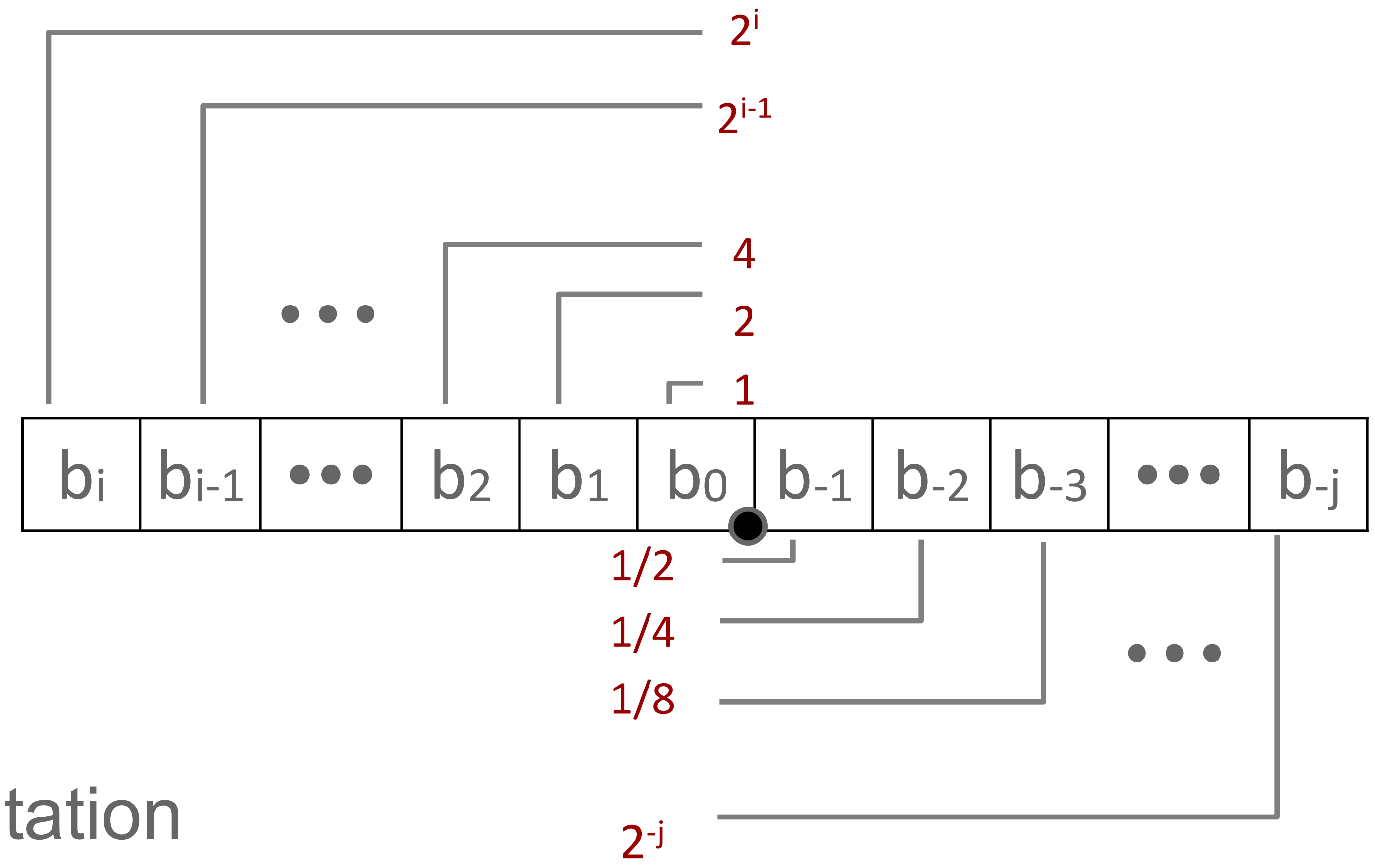
Today: Floating Point

- Background: Fractional binary numbers
- IEEE floating point standard: Definition
- Example and properties
- Rounding, addition, multiplication
- Floating point in C
- Summary

Fractional binary numbers

- What is 1011.101_2?

Fractional Binary Numbers



- Representation
  - Bits to right of “binary point” represent fractional powers of 2
  - Represents rational number: $\sum_{k=-j}^{i} b_k \times 2^k$

Fractional Binary Numbers: Examples

- Value / Representation

Value		Representation	
5 3/4	= 23/4	101.11_2	= 4 + 1 + 1/2 + 1/4
2 7/8	= 23/8	10.111_2	= 2 + 1/2 + 1/4 + 1/8
1 7/16	= 23/16	1.0111_2	= 1 + 1/4 + 1/8 + 1/16

- Observations
  - Divide by 2 by shifting right (unsigned)
  - Multiply by 2 by shifting left
  - Numbers of form $0.111111\ldots_2$ are just below 1.0
    - $1/2 + 1/4 + 1/8 + \ldots + 1/2^i + \ldots \rightarrow 1.0$
    - Use notation $1.0 - \varepsilon$

Representable Numbers

- Limitation #1
  - Can only exactly represent numbers of the form $x/2^k$
    - Other rational numbers have repeating bit representations
  - Value Representation
    - 1/3 `0.0101010101[01]`…2
    - 1/5 `0.001100110011[0011]`…2
    - 1/10 `0.0001100110011[0011]`…2

- Limitation #2
  - Just one setting of binary point within the *w* bits
    - Limited range of numbers (very small values? very large?)

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IEEE Floating Point

- IEEE Standard 754
  - Established in 1985 as uniform standard for floating point arithmetic
    - Before that, many idiosyncratic formats
  - Supported by all major CPUs
  - Some CPUs don't implement IEEE 754 in full
    e.g., early GPUs, Cell BE processor

- Driven by numerical concerns
  - Nice standards for rounding, overflow, underflow
  - Hard to make fast in hardware
    - Numerical analysts predominated over hardware designers in defining standard

Floating Point Representation

Example:
$15213_{10} = (-1)^0 \times 1.1101101101101_2 \times 2^{13}$

- Numerical Form:

$$(-1)^s\ M\ 2^E$$

  - Sign bit s determines whether number is negative or positive
  - Significand M normally a fractional value in range [1.0,2.0).
  - Exponent E weights value by power of two
- Encoding
  - MSB s is sign bit s
  - exp field encodes E (but is not equal to E)
  - frac field encodes M (but is not equal to M)

s	exp	frac

Precision options

- Single precision: 32 bits
  $\approx$ 7 decimal digits, $10^{\pm 38}$

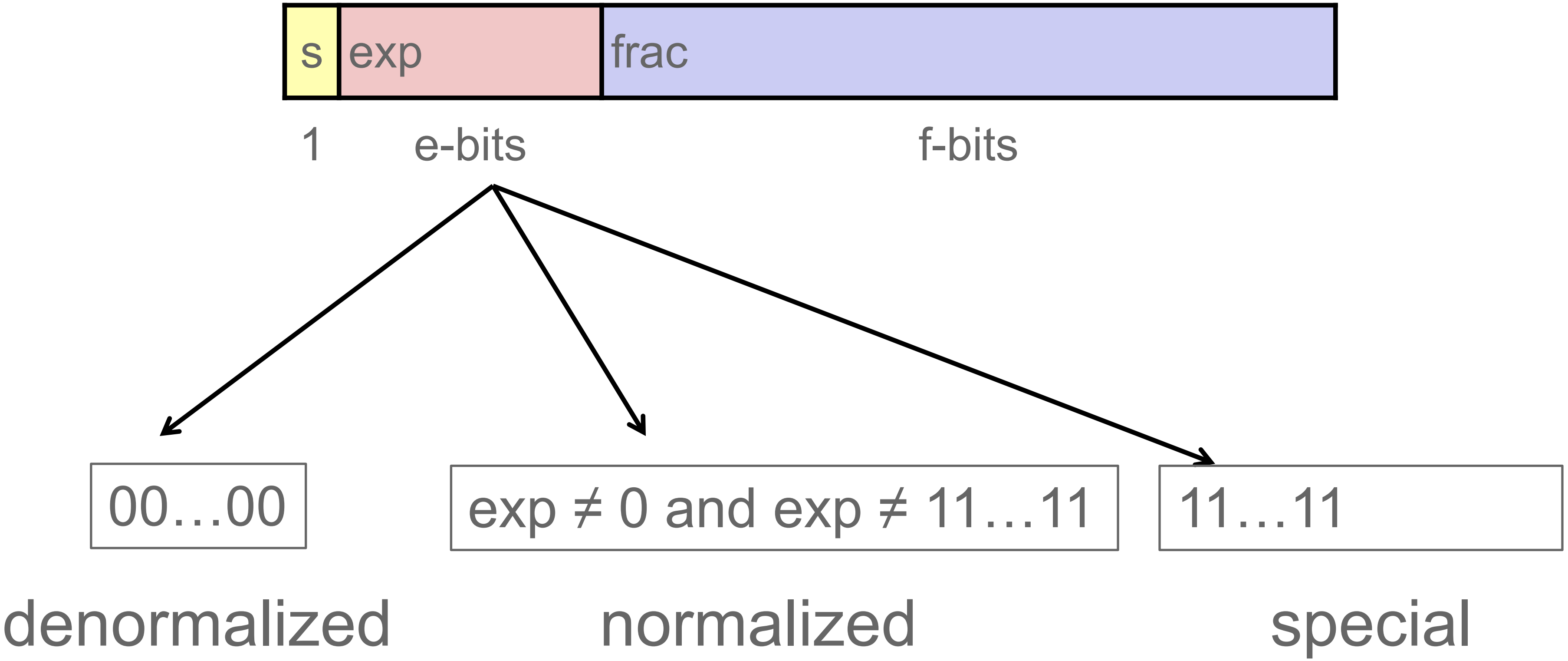
s	exp	frac
1	8-bits	23-bits

- Double precision: 64 bits
  $\approx$ 16 decimal digits, $10^{\pm 308}$

s	exp	frac
1	11-bits	52-bits

- Other formats: half precision, quad precision

Three "kinds" of floating point numbers



"Normalized" Values

$v = (-1)^s M\, 2^E$

- When: **exp** ≠ 000…0 and **exp** ≠ 111…1

- Exponent coded as a biased value: $E = \mathbf{exp} - \text{Bias}$
  - exp: unsigned value of exp field
  - Bias = $2^{k-1} - 1$, where k is number of exponent bits
    - Single precision: 127 (**exp**: 1…254, E: -126…127)
    - Double precision: 1023 (**exp**: 1…2046, E: -1022…1023)

- Significand coded with implied leading 1: $M = 1.xxx{\ldots}x_2$
  - xxx…x: bits of frac field
  - Minimum when **frac**=000…0 (M = 1.0)
  - Maximum when **frac**=111…1 (M = 2.0 – ε)
  - Get extra leading bit for "free"

Normalized Encoding Example

$v = (-1)^s \, M \, 2^E$
$E = \texttt{exp} - \text{Bias}$

- Value: `float F = 15213.0;`
  - $15213_{10} = 11101101101101_2$
    $= 1.1101101101101_2 \times 2^{13}$

- Significand
  M = `1.1101101101101`$_2$
  frac= `11011011011010000000000`$_2$

- Exponent
  E = 13
  $Bias$ = 127
  exp = 140 = `10001100`$_2$

- Result:

s	exp	frac
0	10001100	11011011011010000000000

Denormalized Values

$v = (-1)^s M \, 2^E$
$E = 1 - \text{Bias}$

- Condition: exp = 000…0

- Exponent value: E = 1 – Bias (instead of **exp** – Bias) (why?)
- Significand coded with implied leading 0: $M = 0.xxx\ldots x_2$
  - **xxx…x**: bits of **frac**
- Cases
  - **exp** = 000…0, **frac** = 000…0
    - Represents zero value
    - Note distinct values: +0 and –0 (why?)
  - **exp** = 000…0, **frac** ≠ 000…0
    - Numbers closest to 0.0
    - Equispaced

Special Values

- Condition: **`exp`** = **`111…1`**

- Case: **`exp`** = **`111…1`**, **`frac`** = **`000…0`**
  - **Represents value ∞ (infinity)**
  - Operation that overflows
  - Both positive and negative
  - E.g., 1.0/0.0 = −1.0/−0.0 = +∞, 1.0/−0.0 = −∞

- Case: **`exp`** = **`111…1`**, **`frac`** ≠ **`000…0`**
  - **Not-a-Number (NaN)**
  - Represents case when no numeric value can be determined
  - E.g., sqrt(–1), ∞ − ∞, ∞ × 0

C float Decoding Example

$v = (-1)^s \, M \, 2^E$

$E = \texttt{exp} - \text{Bias}$

$\text{Bias} = 2^{k-1} - 1 = 127$

float: `0xC0A00000`

binary:

E =

S =

M =

$v = (-1)^s \, M \, 2^E =$

Hex	Decimal	Binary
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
A	10	1010
B	11	1011
C	12	1100
D	13	1101
E	14	1110
F	15	1111

C float Decoding Example #1

$v = (-1)^s M 2^E$

$E = \texttt{exp} - \text{Bias}$

float: `0xC0A00000`

binary: `1100 0000 1010 0000 0000 0000 0000 0000`

1	1000 0001	010 0000 0000 0000 0000 0000
1	8-bits	23-bits

E =

S =

M = `1.`

$v = (-1)^s M 2^E =$

Hex	Decimal	Binary
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
A	10	1010
B	11	1011
C	12	1100
D	13	1101
E	14	1110
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C float Decoding Example #1

$v = (-1)^s M\ 2^E$

$E = \texttt{exp} - \text{Bias}$

$\text{Bias} = 2^{k-1} - 1 = 127$

float: `0xC0A00000`

binary: `1100 0000 1010 0000 0000 0000 0000 0000`

1	1000 0001	010 0000 0000 0000 0000 0000
1	8-bits	23-bits

$E = \texttt{exp} - \text{Bias} = 129 - 127 = 2$ (decimal)

S = 1 -> negative number

M = `1.010 0000 0000 0000 0000 0000`

= `1 + 1/4 = 1.25`

$v = (-1)^s M\ 2^E = (-1)^1 * 1.25 * 2^2 = -5$

Hex	Decimal	Binary
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
A	10	1010
B	11	1011
C	12	1100
D	13	1101
E	14	1110
F	15	1111

C float Decoding Example #2

$v = (-1)^s M 2^E$

$E = 1 - \text{Bias}$

float: `0x001C0000`

binary: 0000 0000 0001 1100 0000 0000 0000 0000

0	0000 0000	001 1100 0000 0000 0000 0000
1	8-bits	23-bits

E =

S =

M = 0.

$v = (-1)^s M 2^E =$

Hex	Decimal	Binary
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
A	10	1010
B	11	1011
C	12	1100
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E	14	1110
F	15	1111

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C float Decoding Example #2

$v = (-1)^s \, M \, 2^E$

$E = 1 - \text{Bias}$

Bias = $2^{k-1} - 1 = 127$

float: 0x001C0000

binary: 0000 0000 0001 1100 0000 0000 0000 0000

0	0000 0000	001 1100 0000 0000 0000 0000
1	8-bits	23-bits

E = 1 – Bias = 1 – 127 = –126 (decimal)

S = 0 -> positive number

M = 0.001 1100 0000 0000 0000 0000

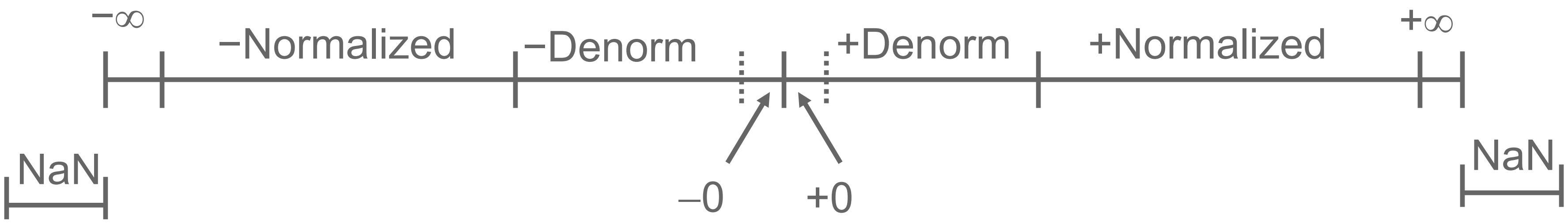
= 1/8 + 1/16 + 1/32 = 7/32 = $7*2^{-5}$

$v = (-1)^s \, M \, 2^E = (-1)^0 * 7*2^{-5} * 2^{-126} = 7*2^{-131}$

$\approx 2.571393892 \times 10^{-39}$

Hex	Decimal	Binary
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
A	10	1010
B	11	1011
C	12	1100
D	13	1101
E	14	1110
F	15	1111

Visualization: Floating Point Encodings



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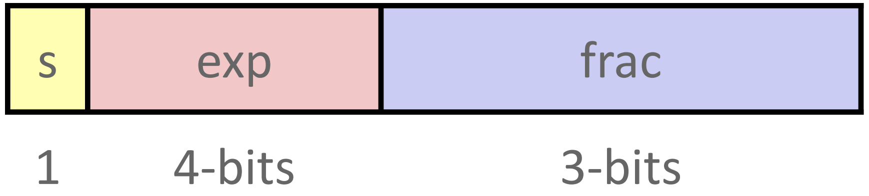
Tiny Floating Point Example



- 8-bit Floating Point Representation
  - the sign bit is in the most significant bit
  - the next four bits are the **`exp`**, with a bias of 7
  - the last three bits are the **`frac`**
- Same general form as IEEE Format
  - normalized, denormalized
  - representation of 0, NaN, infinity

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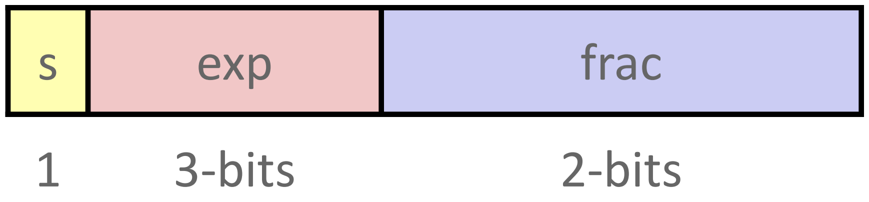
Dynamic Range (s=0 only)

$v = (-1)^s M 2^E$
norm: E = **exp** – Bias
denorm: E = 1 – Bias

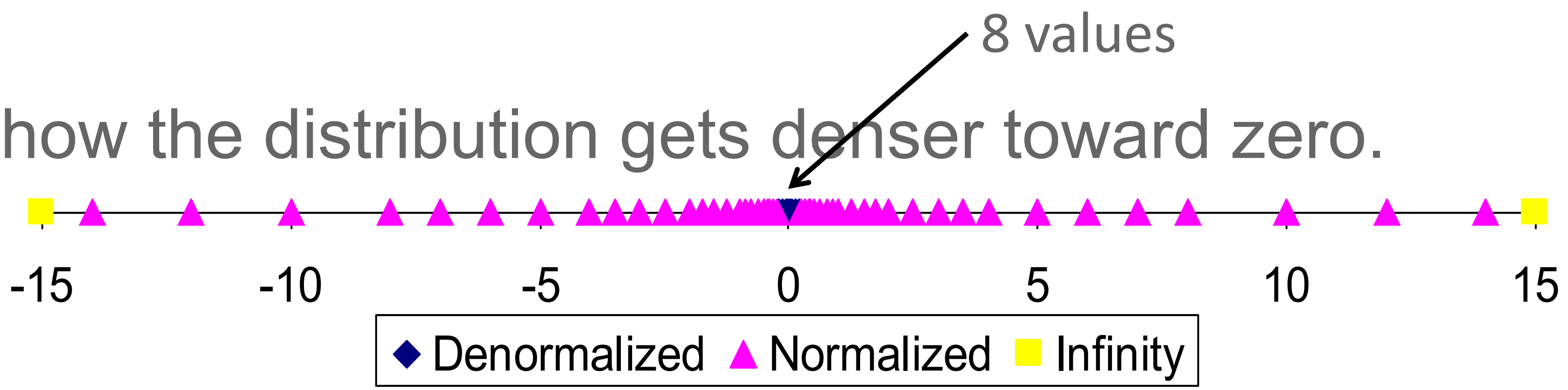
	s	exp	frac	E	Value	
Denormalized numbers	0	0000	000	-6	0	
	0	0000	001	-6	1/8*1/64 = 1/512	closest to zero
	0	0000	010	-6	2/8*1/64 = 2/512	$(-1)^0(0+1/4)*2^{-6}$
	…					
	0	0000	110	-6	6/8*1/64 = 6/512	
	0	0000	111	-6	7/8*1/64 = 7/512	largest denorm
Normalized numbers	0	0001	000	-6	8/8*1/64 = 8/512	smallest norm
	0	0001	001	-6	9/8*1/64 = 9/512	$(-1)^0(1+1/8)*2^{-6}$
	…					
	0	0110	110	-1	14/8*1/2 = 14/16	
	0	0110	111	-1	15/8*1/2 = 15/16	closest to 1 below
	0	0111	000	0	8/8*1 = 1	
	0	0111	001	0	9/8*1 = 9/8	closest to 1 above
	0	0111	010	0	10/8*1 = 10/8	
	…					
	0	1110	110	7	14/8*128 = 224	
	0	1110	111	7	15/8*128 = 240	largest norm
	0	1111	000	n/a	inf	

Distribution of Values

- 6-bit IEEE-like format
  - e = 3 exponent bits
  - f = 2 fraction bits
  - Bias is $2^{3-1}-1 = 3$

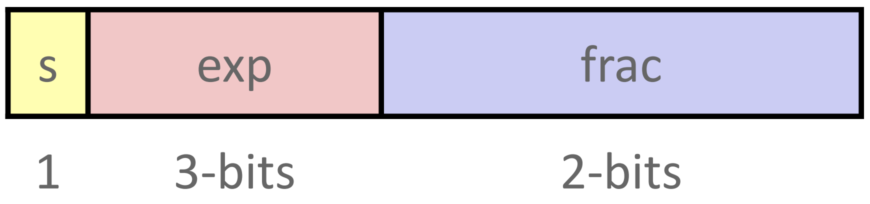


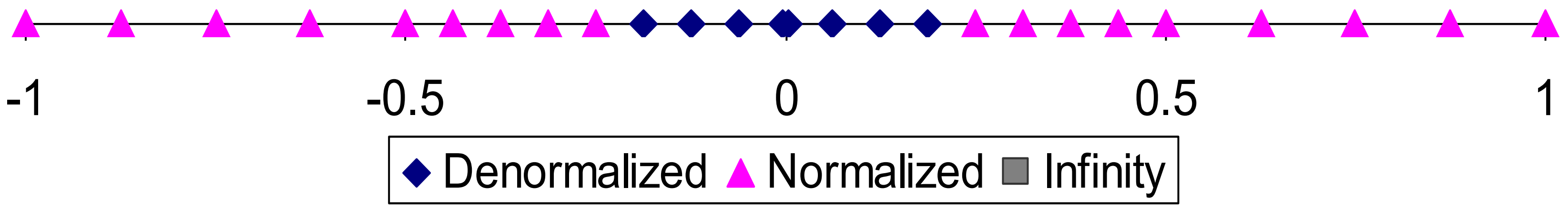
- Notice how the distribution gets denser toward zero.



Distribution of Values (close-up view)

- 6-bit IEEE-like format
  - e = 3 exponent bits
  - f = 2 fraction bits
  - Bias is 3





Special Properties of the IEEE Encoding

- FP Zero Same as Integer Zero
  - All bits = 0

- Can (Almost) Use Unsigned Integer Comparison
  - Must first compare sign bits
  - Must consider −0 = 0
  - NaNs problematic
    - Will be greater than any other values
    - What should comparison yield? The answer is complicated.
  - Otherwise OK
    - Denorm vs. normalized
    - Normalized vs. infinity

Floating Point Operations: Basic Idea

- $x +_f y = \text{Round}(x + y)$

- $x \times_f y = \text{Round}(x \times y)$

- Basic idea
  - First **compute exact result**
  - Make it fit into desired precision
    - Possibly overflow if exponent too large
    - Possibly **round to fit into** `frac`

Rounding

- Rounding Modes (illustrate with $ rounding)

	\$1.40	\$1.60	\$1.50	\$2.50	–\$1.50
• Towards zero	\$1	\$1	\$1	\$2	–\$1
• Round down (−∞)	\$1	\$1	\$1	\$2	–\$2
• Round up (+∞)	\$2	\$2	\$2	\$3	–\$1
• Nearest Even* (default)		\$1	\$2	\$2	\$2
	–\$2				

*Round to nearest, but if half-way in-between then round to nearest even

Closer Look at Round-To-Even

- Default Rounding Mode
  - Hard to get any other kind without dropping into assembly
    - C99 has support for rounding mode management
  - All others are statistically biased
    - Sum of set of positive numbers will consistently be over- or under-estimated
- Applying to Other Decimal Places / Bit Positions
  - When exactly halfway between two possible values
    - Round so that least significant digit is even
  - E.g., round to nearest hundredth

7.8949999	7.89	(Less than half way)
7.8950001	7.90	(Greater than half way)
7.8950000	7.90	(Half way—round up)
7.8850000	7.88	(Half way—round down)

Rounding Binary Numbers

- Binary Fractional Numbers
  - “Even” when least significant bit is `0`
  - “Half way” when bits to right of rounding position = `100…`$_2$

- Examples
  - Round to nearest 1/4 (2 bits right of binary point)

Value	Binary	Rounded	Action	Rounded Value
2 3/32	10.00011_2	10.00_2	(<1/2—down)	2
2 3/16	10.00110_2	10.01_2	(>1/2—up)	2 1/4
2 7/8	10.11100_2	11.00_2	(1/2—up)	3
2 5/8	10.10100_2	10.10_2	(1/2—down)	2 1/2

Rounding

1.BBGRXXX

Guard bit: LSB of result

Round bit: 1st bit removed

Sticky bit: OR of remaining bits

- Round up conditions
  - Round = 1, Sticky = 1 → > 0.5
  - Guard = 1, Round = 1, Sticky = 0 → Round to even

Fraction	GRS	Incr?	Rounded
1.0000000	000	N	1.000
1.1010000	100	N	1.101
1.0001000	010	N	1.000
1.0011000	110	Y	1.010
1.0001010	011	Y	1.001
1.1111100	111	Y	10.000

FP Multiplication

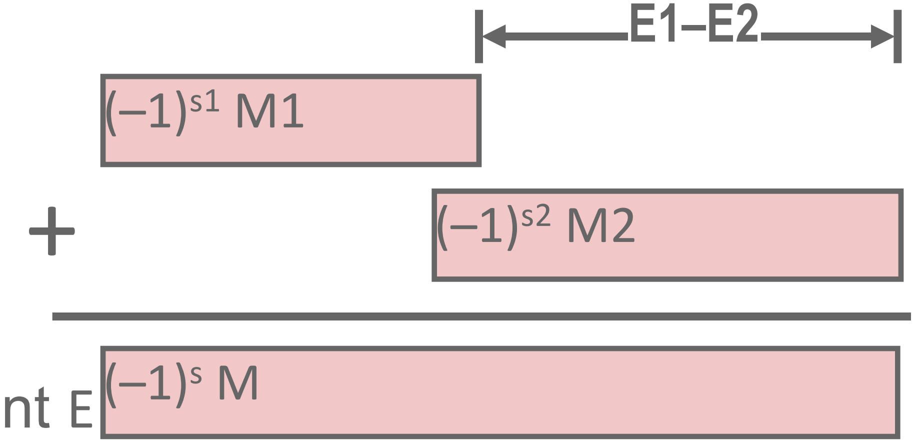
- $(-1)^{s1}$ M1 2^{E1} x $(-1)^{s2}$ M2 2^{E2}
- Exact Result: $(-1)^{s}$ M 2^{E}
  - Sign s: s1 ^ s2
  - Significand M: M1 x M2
  - Exponent E: E1 + E2
- Fixing
  - If M ≥ 2, shift M right, increment E
  - If E out of range, overflow
  - Round M to fit `frac` precision
- Implementation
  - Biggest chore is multiplying significands

4 bit significand: 1.010*2^2 x 1.110*2^3 = 10.0011*2^5
= 1.00011*2^6 = 1.001*2^6

Floating Point Addition

- $(-1)^{s1}\ M1\ \ 2^{E1}\ \ +\ \ (-1)^{s2}\ M2\ \ 2^{E2}$
  - Assume E1 > E2
- Exact Result: $(-1)^{s}\ M\ \ 2^{E}$
  - Sign s, significand M:
    - Result of signed align & add
  - Exponent E: E1
- Fixing
  - If M ≥ 2, shift M right, increment E
  - if M < 1, shift M left k positions, decrement E by k
  - Overflow if E out of range
  - Round M to fit `frac` precision

Get binary points lined up



$1.010*2^2 + 1.110*2^3 = (0.1010 + 1.1100)*2^3$
$= 10.0110 * 2^3 = 1.00110 * 2^4 = 1.010 * 2^4$

Mathematical Properties of FP Add

- Compare to those of Abelian Group
  - Closed under addition? **Yes**
    - But may generate infinity or NaN
  - Commutative? **Yes**
  - Associative? **No**
    - Overflow and inexactness of rounding
    - `(3.14+1e10)-1e10 = 0, 3.14+(1e10-1e10) = 3.14`
  - 0 is additive identity? **Yes**
  - Every element has additive inverse? **Almost**
    - Yes, except for infinities & NaNs
- Monotonicity
  - $a \geq b \Rightarrow a+c \geq b+c$? **Almost**
    - Except for infinities & NaNs

Mathematical Properties of FP Mult

- Compare to Commutative Ring
  - Closed under multiplication? **Yes**
    - But may generate infinity or NaN
  - Multiplication Commutative? **Yes**
  - Multiplication is Associative? **No**
    - Possibility of overflow, inexactness of rounding
    - Ex: `(1e20*1e20)*1e-20= inf`, `1e20*(1e20*1e-20)= 1e20`
  - 1 is multiplicative identity? **Yes**
  - Multiplication distributes over addition? **No**
    - Possibility of overflow, inexactness of rounding
    - `1e20*(1e20-1e20)= 0.0, 1e20*1e20 – 1e20*1e20 = NaN`
- Monotonicity
  - $a \geq b$ & $c \geq 0 \Rightarrow a * c \geq b * c$? **Almost**
    - Except for infinities & NaNs